

# On the Hitchin morphism in positive characteristic

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## Abstract

Let  $X$  be a smooth projective curve over a field of characteristic  $p > 0$ . We show that the Hitchin morphism, which associates to a Higgs bundle its characteristic polynomial, has a non-trivial deformation over the affine line. This deformation is constructed by considering the moduli stack of  $t$ -connections on vector bundles on  $X$  and an analogue of the  $p$ -curvature, and by observing that the associated characteristic polynomial is, in a suitable sense, a  $p^{\text{th}}$ -power.

## 1 Introduction

Let  $X$  be a smooth projective curve over an algebraically closed field  $k$  and let  $\omega_X$  be its canonical line bundle. The Hitchin morphism associates to a rank  $r$  vector bundle  $E$  of degree zero and a Higgs field  $\phi : E \rightarrow E \otimes \omega_X$  its characteristic polynomial, denoted by  $H(E, \phi)$ , which lies in the affine space  $W = \bigoplus_{i=1}^r H^0(X, \omega_X^i)$ . Thus one gets a morphism

$$H : \mathcal{Higgs}(r, X) \longrightarrow W$$

from the moduli stack of Higgs bundles to  $W$ , which becomes universally closed, when restricted to the substack of semi-stable Higgs bundles [N] [F]. Moreover, if  $k = \mathbb{C}$ , it is shown [H] that  $H$  is an algebraically completely integrable system.

In this note we show that the Hitchin morphism  $H$  has a non-trivial deformation over the affine line if the characteristic of  $k$  is  $p > 0$ . More precisely, we consider the moduli stack  $\mathcal{C}(r, X)$  of  $t$ -connections  $\nabla_t$  on rank  $r$  vector bundles  $E$  over  $X$  with  $t \in k$ . A  $t$ -connection  $\nabla_t$  can be thought of as an “interpolating” object between a Higgs field ( $t = 0$ ) and a connection ( $t = 1$ ). Now one associates to  $\nabla_t$  a suitable analogue of the  $p$ -curvature and it turns out (Proposition 3.2) that its characteristic polynomial is a  $p^{\text{th}}$ -power. This fact entails the existence of a morphism over  $\mathbb{A}^1$

$$H : \mathcal{C}(r, X) \longrightarrow W \times \mathbb{A}^1$$

which restricts ( $t = 0$ ) to the Hitchin morphism for Higgs bundles. Finally we prove that the restriction of  $H$  to the substack of semi-stable nilpotent  $t$ -connections is universally closed. This provides a non-trivial deformation of the semi-stable locus of the global nilpotent cone. This result can be considered as an analogue of Simpson’s result which says that the moduli space of representations of the fundamental group has the homotopy type of the global nilpotent cone.

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## 2 Notations

We will denote by  $S$  a  $k$ -scheme with  $k$  a field of characteristic  $p > 0$  and by  $\mathcal{X}$  (resp.  $X$ ) a smooth projective connected  $S$ -curve (resp.  $k$ -curve).

### 2.1

We denote by  $f_S : S \rightarrow S$  the absolute Frobenius (which is topologically the identity and the  $p^{\text{th}}$  power on functions). We denote by  $S^{(p)}$  the inverse image of  $S$  by the Frobenius  $f_k$  of  $k$ . There exists a unique commutative diagram

$$\begin{array}{ccccc} S & \xrightarrow{F_S} & S^{(p)} & \xrightarrow{\pi_S} & S \\ & \searrow & \downarrow & & \downarrow \\ & & \text{Spec}(k) & \xrightarrow{f_k} & \text{Spec}(k) \end{array}$$

such that  $\pi_S \circ F_S = f_S$ . The  $k$ -morphism  $F_S$  is called the relative Frobenius. If  $S$  is defined by equations  $f_j = \sum_I a_{I,j} x^I$  in an affine space  $\mathbb{A}^N$  with coordinates  $(x_i)_{1 \leq i \leq N}$  and  $a_{I,j} \in k$ , then  $S^{(p)}$  is the subscheme of  $\mathbb{A}^N$  defined by the equations  $f_j^{[p]} = \sum_I a_{I,j}^p x^I$ . In this case,  $F_S : S \rightarrow S^{(p)}$  is given by  $x_i \mapsto x_i^p$ .

### 2.2

In the relative case  $\mathcal{X} \rightarrow S$ , we denote by  $\mathcal{X}^{(p)}$  the pullback of  $\mathcal{X}$  by  $f_S$ . Assuming  $S = \text{Spec}(R)$ , with  $R$  a  $k$ -algebra, and  $\mathcal{X}$  given by equations  $f_j = \sum_I a_{I,j} x^I$  in an affine space  $\mathbb{A}_R^N$  with coordinates  $(x_i)_{1 \leq i \leq N}$  and  $a_{I,j} \in R$ , then  $\mathcal{X}^{(p)}$  is the subscheme of  $\mathbb{A}_R^N$  defined by the equations  $f_j^{[p]} = \sum_I a_{I,j}^p x^I$ . The  $S$ -morphism  $F_{\mathcal{X}/S} : \mathcal{X} \rightarrow \mathcal{X}^{(p)}$  is given by  $x_i \mapsto x_i^p$ .

### 2.3

In the case  $\mathcal{X} = X \times S$ , the commutative diagram

$$\begin{array}{ccc} X^{(p)} \times S & \xrightarrow{(\pi_X, f_S)} & X \times S \\ \downarrow & & \downarrow \\ S & \xrightarrow{f_S} & S \end{array}$$

defines a morphism  $X^{(p)} \times S \rightarrow (X \times S)^{(p)} = \mathcal{X}^{(p)}$ , which is an isomorphism thanks to the local description given in (2.1) and (2.2). Therefore, we will identify the  $S$ -schemes  $\mathcal{X}^{(p)}$  and  $X^{(p)} \times S$  and accordingly the relative Frobenius morphisms  $F_{\mathcal{X}/S}$  and  $F_X \times \text{Id}_S$ .

## 3 The stack of bundles with connection

Let  $\mathcal{X}$  be a smooth projective curve over a basis  $S$  of characteristic  $p$ .

### 3.1

We denote by  $\mathcal{P}_{\mathcal{X}/S}^1$  the sheaf of principal parts of degree  $\leq 1$ , namely the structural sheaf of the second order infinitesimal neighborhood  $\Delta^{(2)}$  of the diagonal in  $\mathcal{X} \times_S \mathcal{X}$  ([G] section 16). The left  $\mathcal{O}_{\mathcal{X}}$ -module structure is defined by the first projection and the right one by the second projection.

Let 1 be the global section of  $\mathcal{P}_{\mathcal{X}/S}^1$  defined by the constant function with value 1 on  $\Delta^{(2)}$ . The restriction to the diagonal gives the exact sequence

$$0 \rightarrow \omega_{\mathcal{X}/S} \rightarrow \mathcal{P}_{\mathcal{X}/S}^1 \xrightarrow{\pi} \mathcal{O}_{\mathcal{X}} \rightarrow 0$$

of left  $\mathcal{O}_{\mathcal{X}}$ -modules. We tensorize this exact sequence with a vector bundle  $\mathcal{E}$  on  $\mathcal{X}$ .

$$(3.1) \quad 0 \rightarrow \omega_{\mathcal{X}/S} \otimes \mathcal{E} \rightarrow \mathcal{P}_{\mathcal{X}/S}^1 \otimes \mathcal{E} \rightarrow \mathcal{E} \rightarrow 0$$

Note that  $\mathcal{P}_{\mathcal{X}/S}^1 \otimes \mathcal{E}$  is the left-module tensor product of the bimodule  $\mathcal{P}_{\mathcal{X}/S}^1$  by the left-module  $\mathcal{E}$ . The following observation is classical (and tautological).

**Lemma 3.1** (i) Let  $\sigma \in \text{Hom}(\mathcal{E}, \mathcal{P}_{\mathcal{X}/S}^1 \otimes \mathcal{E})$  be a splitting of (3.1). Then the morphism of sheaves  $e \mapsto 1.e - \sigma(e)$  takes its values in  $\omega_{\mathcal{X}/S} \otimes \mathcal{E}$  and is a connection  $\nabla_{\sigma}$ .  
(ii) The map  $\sigma \mapsto \nabla_{\sigma}$  from splittings of (3.1) to connections on  $\mathcal{E}$  is bijective.

The functoriality of the sheaf of principal parts allows us to define the pull-back of a connection by a base change  $S' \rightarrow S$  (which is a connection on the  $S'$ -curve  $\mathcal{X}' = \mathcal{X} \times_S S'$ ).

## 3.2

Let  $t \in H^0(S, \mathcal{O})$  be a function on  $S$ . We also denote by  $t$  its pull-back to  $\mathcal{X}$ . We define the twisted sheaf of principal part  $\mathcal{P}_{\mathcal{X}/S}^t$  as the kernel of

$$\begin{cases} \mathcal{P}_{\mathcal{X}/S}^1 \oplus \mathcal{O}_{\mathcal{X}} & \rightarrow & \mathcal{O}_{\mathcal{X}} \\ (p, e) & \mapsto & \pi(p) - te \end{cases}$$

The sheaf  $\mathcal{P}_{\mathcal{X}/S}^t$  is an  $\mathcal{O}_{\mathcal{X}}$ -bimodule. The canonical exact sequence

$$(3.2) \quad 0 \rightarrow \omega_{\mathcal{X}/S} \otimes \mathcal{E} \rightarrow \mathcal{P}_{\mathcal{X}/S}^t \otimes \mathcal{E} \rightarrow \mathcal{E} \rightarrow 0$$

is the pullback of (3.1) by the scalar endomorphism of  $\mathcal{E}$  defined by  $t$ .

Recall that a  $t$ -connection on the vector bundle  $\mathcal{E}$  on  $\mathcal{X}$  is a morphism of  $\mathcal{O}_S$ -modules

$$\nabla_t : \mathcal{E} \rightarrow \omega_{\mathcal{X}/S} \otimes \mathcal{E}$$

satisfying the (twisted) Leibniz rule

$$\nabla_t(fe) = tdf \otimes e + f\nabla_t(e)$$

where  $f$  and  $e$  are local sections of  $\mathcal{O}_{\mathcal{X}}$  and  $\mathcal{E}$ , respectively. We say that the pair  $(\mathcal{E}, \nabla_t)$  is a  $t$ -bundle. For instance a 0-connection is simply a Higgs field and, if  $t$  is invertible,  $\nabla_t/t$  is a connection. Conversely, given a connection  $\nabla$  on  $\mathcal{E}$ , then  $\nabla_t := t\nabla$  is a  $t$ -connection.

The twisted version of Lemma 3.1 is

**Lemma 3.2** (i) Let  $\sigma \in \text{Hom}(\mathcal{E}, \mathcal{P}_{\mathcal{X}/S}^t \otimes \mathcal{E})$  be a splitting of (3.2). Then the morphism of sheaves  $e \mapsto 1.e - \sigma(e)$  takes its values in  $\omega_{\mathcal{X}/S} \otimes \mathcal{E}$  and is a  $t$ -connection  $\nabla_{\sigma}$ .  
(ii) The map  $\sigma \mapsto \nabla_{\sigma}$  from splittings of (3.2) to  $t$ -connections on  $\mathcal{E}$  is bijective.

### 3.3

Recall that  $X$  is a smooth projective curve over  $k$ . Let  $S$  be a  $t$ -scheme, i.e., endowed with a global function  $t$ . Let  $\mathcal{C}^*(S)$  be the category whose objects are pairs  $(\mathcal{E}, \nabla_t)$  where  $\mathcal{E}$  is a degree zero vector bundle on  $X_S = X \times S$  and  $\nabla_t$  is a  $t$ -connection on  $\mathcal{E}$  and whose morphisms are isomorphisms commuting with the  $t$ -connections (we still say flat morphisms). These categories with the obvious inverse image functor define a fibred category over  $\text{Aff}/\mathbb{A}^1$  which is obviously a stack, denoted by  $\mathcal{C}(r, X)$ . Let  $\mathcal{M}(r, X)$  denote the stack of rank  $r$  vector bundles of degree zero over  $X$ .

**Proposition 3.1** *The forgetful morphism  $\mathcal{C}(r, X) \rightarrow \mathcal{M}(r, X) \times \mathbb{A}^1$  is representable.*

*Proof:* Let  $(\mathcal{E}, t)$  be an object of  $\text{Hom}_{\mathbb{A}^1}(S, \mathcal{M}(r, X) \times \mathbb{A}^1)$ . Let  $\omega$  be the dualizing sheaf of  $X_S \rightarrow S$  (the pull-back of the canonical bundle of  $X$ ) and  $\mathcal{V}$  be the dual of  $\mathcal{H}om(\mathcal{E}, \mathcal{P}_{X_S/S}^t \otimes \mathcal{E}) \otimes \omega^{-1}$ . Let us consider the commutative diagram with cartesian square

$$\begin{array}{ccccc} & X_{S'} & \xrightarrow{g} & X_S & \\ p' \downarrow & & & & \downarrow p \\ & S' & \xrightarrow{f} & S & \\ t' \searrow & & & & \swarrow t \\ & \mathbb{A}^1 & & & \end{array}$$

The category of triples  $(\alpha, \mathcal{E}', \nabla_{t'})$  where  $\alpha'$  is an isomorphism  $\mathcal{E}' \xrightarrow{\sim} g^*\mathcal{E}$  and  $\nabla'$  a  $t'$ -connection is equivalent with the subset (thought of as a discrete category) of

$$H(S') = \text{Hom}(g^*\mathcal{E}, \mathcal{P}_{X_{S'}/S'}^{t'} \otimes g^*\mathcal{E})$$

mapping to the identity of  $g^*\mathcal{E}$ . The pull-back  $\omega'$  of  $\omega$  is the dualizing sheaf of  $X_{S'}$ . Therefore

$$H(S') = H^0(p'_*\mathcal{H}om(g^*\mathcal{V}, \omega')) = \text{Hom}(R^1p'_*g^*\mathcal{V}, \mathcal{O}_{S'})$$

by duality. By base change theory,

$$R^1p'_*g^*\mathcal{V} = f^*R^1p_*\mathcal{V}$$

showing the canonical identification

$$H(S') = \text{Hom}_S(S', V(R^1p_*\mathcal{V})).$$

In the same way,  $S' \rightarrow \text{Hom}(g^*\mathcal{E}, g^*\mathcal{E})$  is representable by  $V(R^1p_*\mathcal{W})$  where  $\mathcal{W}$  is the dual of  $\mathcal{H}om(\mathcal{E}, \mathcal{E}) \otimes \omega^{-1}$ . The functorial morphism induces an  $S$ -morphism

$$V(R^1p_*\mathcal{V}) \rightarrow V(R^1p_*\mathcal{W})$$

The identity of  $\mathcal{E}$  defines a section of  $V(R^1p_*\mathcal{W}) \rightarrow S$  and our  $S$ -category is represented by the fibred product

$$S \times_{V(R^1p_*\mathcal{W})} V(R^1p_*\mathcal{V}).$$

■

Therefore, by general theory [LM], one gets

**Corollary 3.1** *The stack  $\mathcal{C}(r, X)$  is algebraic, locally of finite type over  $\mathbb{A}^1$ .*

### 3.4

Let  $\nabla$  be a connection on the rank  $r$  vector bundle  $\mathcal{E}$  over  $\mathcal{X}$ . Recall that the  $p$ -curvature  $\Psi(\nabla)$  of  $\nabla$  is the mapping ([K] section 5)

$$\Psi(\nabla) : \begin{cases} T_{\mathcal{X}/S} & \rightarrow \mathcal{E}nd_S(\mathcal{E}) \\ D & \mapsto (\nabla(D))^p - \nabla(D^p) \end{cases}$$

Recall that the relative tangent sheaf  $T_{\mathcal{X}/S}$  is a sheaf of restricted  $p$ -Lie algebras.

**Remark 3.1** Notice that the curvature is the obstruction for the connection to define a morphism of Lie algebra and that the  $p$ -curvature is the obstruction for an integrable connection to define a morphism of restricted  $p$ -Lie algebras.

The additive morphism  $\Psi(\nabla)$  is  $\mathcal{O}_{\mathcal{X}}$ -semi-linear, i.e., additive and  $\Psi(\nabla)(gD) = g^p \Psi(\nabla)(D)$  for  $g$  and  $D$  local sections of  $\mathcal{O}_{\mathcal{X}}$  and  $T_{\mathcal{X}/S}$  respectively ([K] Proposition 5.2). Therefore it defines an  $\mathcal{O}_{\mathcal{X}}$ -linear morphism

$$\Psi(\nabla) : T_{\mathcal{X}/S} \rightarrow f_{\mathcal{X}*} \mathcal{E}nd_S(\mathcal{E})$$

where  $f_{\mathcal{X}}$  is the absolute Frobenius of  $\mathcal{X}$ . We still denote by  $\Psi(\nabla)$  the corresponding section in  $\text{Hom}(\mathcal{E}, \mathcal{E} \otimes f_{\mathcal{X}}^* \omega_{\mathcal{X}/S})$  obtained by adjunction.

### 3.5

From now on we are interested in the case  $\mathcal{X} = X \times S$ . Let  $q$  be the first projection. Since  $f_{\mathcal{X}}^* \omega_{\mathcal{X}/S} = q^* \omega_X^p$ ,  $\Psi(\nabla)$  can be thought of as a point of  $\text{Hom}(\mathcal{E}, \mathcal{E} \otimes q^* \omega_X^p)$ . Let  $\nabla_t$  be a  $t$ -connection on  $\mathcal{E}$  with  $t \in H^0(S, \mathcal{O})$ . If  $t \in \mathbb{G}_m(S)$ , then  $\nabla_t/t$  is a connection on  $\mathcal{E}$  and we have, for any local section  $D$  of  $T_X$

$$t^p \Psi(\nabla_t/t)(D) = (\nabla_t(D))^p - t^{p-1} \nabla_t(D^p) \in \mathcal{E}nd_S(\mathcal{E}).$$

**Definition 3.1** The  $p$ -curvature  $\Psi_t(\nabla_t)$  of  $\nabla_t$  is the additive morphism

$$\Psi_t(\nabla_t) : \begin{cases} T_{\mathcal{X}/S} & \rightarrow \mathcal{E}nd_S(\mathcal{E}) \\ D & \mapsto (\nabla_t(D))^p - t^{p-1} \nabla_t(D^p) \end{cases}$$

**Lemma 3.3** The  $p$ -curvature of  $\nabla_t$  is  $\mathcal{O}_{\mathcal{X}}$ -semi-linear.

*Proof:* We can adapt the proof of Proposition 5.2 [K] to the  $t$ -connection  $\nabla_t$  taking into account that the commutator of  $\nabla_t(D)$  and  $g$  in  $\mathcal{E}nd_S(\mathcal{E})$  is  $tD(g)$  (see formula (5.4.2) of [K]). ■

As in (3.4) we still denote by  $\Psi_t(\nabla_t)$  the corresponding element in  $\text{Hom}(\mathcal{E}, \mathcal{E} \otimes q^* \omega_X^p)$ .

### 3.6

Let us denote by  $V$  the affine variety  $V = \bigoplus_{i=1}^r H^0(X, \omega_X^{p_i})$  and by  $V(S) = V \times S$ .

**Definition 3.2** Let  $\nabla_t$  be a  $t$ -connection on  $\mathcal{E}$ . We denote by  $\text{Char}(\nabla_t)$  the point of  $V(S)$  defined by the coefficients of the characteristic polynomial of the morphism  $\Psi_t(\nabla_t) : \mathcal{E} \rightarrow \mathcal{E} \otimes q^* \omega_X^p$ .

**Remark 3.2** The functor  $\nabla_t \mapsto (\text{Char}(\nabla_t), t)$  defines a morphism of stacks

$$\underline{\text{Char}} : \mathcal{C}(r, X) \rightarrow V \times \mathbb{A}^1$$

Given a Higgs bundle  $(E, \nabla_0)$  over  $X$ , we observe that

$$\text{Char}(\nabla_0) = (H_0(\nabla_0))^p,$$

where  $H_0$  is the Hitchin morphism as defined in [H]. As will be shown in the next section,  $\text{Char}(\nabla_t) \in V$  remains a  $p^{\text{th}}$  power when considering, more generally,  $t$ -connections  $\nabla_t$ .

### 3.7

Let us denote by  $W$  the affine variety  $W = \bigoplus_{i=1}^r H^0(X, \omega_X^i)$ . The absolute Frobenius morphism  $f_X : X \rightarrow X$  induces an injective  $p$ -linear map  $f_X^* : W \hookrightarrow V$ .

**Proposition 3.2** *There exists a unique morphism  $H$  over  $\mathbb{A}^1$*

$$H : \mathcal{C}(r, X) \rightarrow W \times \mathbb{A}^1$$

making the diagram

$$\begin{array}{ccc} \mathcal{C}(r, X) & & \\ \downarrow H & \searrow \text{Char} & \\ W \times \mathbb{A}^1 & \hookrightarrow & V \times \mathbb{A}^1 \end{array}$$

commutative and such that the restriction  $H_0$  of  $H$  to the fibre over  $0 \in \mathbb{A}^1$  coincides with the Hitchin morphism.

*Proof:* Let  $(\mathcal{E}, \nabla_t)$  be a  $t$ -bundle over  $X \times S$ . We can assume  $S = \text{Spec}(R)$ , with  $R$  integral since the stack  $\mathcal{C}(r, X)$  is smooth. To show the proposition, it will be enough to show that  $\text{Char}(\nabla_t) \in W(S) \hookrightarrow V(S)$  or, equivalently, that the global sections  $\text{Char}(\nabla_t) = (s_i) \in \bigoplus_i H^0(X, \omega_X^{ip}) \otimes R$  descend to  $X^{(p)} \times S$  by the relative Frobenius  $F_X \times \text{Id} : X \times S \rightarrow X^{(p)} \times S$ . By Cartier's theorem ([K] Theorem 5.1) it suffices to check that for all  $i$ ,  $\nabla_t^{\text{can}} s_i = 0$ , where  $\nabla_t^{\text{can}}$  is the canonical connection on  $\omega_X^{ip} = F_{X^{(p)}}^* \omega_{X^{(p)}}^i$ . The question is local on  $X$ . Let  $z$  be a local coordinate near a geometric point  $x$  on  $X$ . We choose trivializations of  $\omega_X$  and  $\mathcal{E}$ . Then  $\nabla_t^{\text{can}}$  is equal to the derivation  $d/dz$ , which we denote by  $\partial$ , and the operator  $\nabla_t(\partial) \in \text{End}(\mathcal{E})$  can be written as

$$(3.3) \quad \nabla_t(\partial) = t\partial + A$$

with  $t \in R$  and  $A$  an  $r \times r$  matrix with entries in the ring  $R[[z]]$  (we work in a completion of the local ring  $\mathcal{O}_{X,x}$ ). We consider the  $p$ -curvature (note that  $\partial^p = \partial$ )

$$(3.4) \quad \Psi_t(\partial) = (t\partial + A)^p$$

as an  $r \times r$  matrix, which we simply denote by  $\Psi_t$ . By  $p$ -linearity of the  $p$ -curvature  $\Psi_t$  and by Cartier's theorem, one just has to show that the derivative

$$\partial \cdot \det(\text{Id} - T\Psi_t) = 0.$$

The following elegant argument is due to J.B. Bost. From (3.3) and (3.4) it is obvious that the commutator of  $\nabla_t(\partial)$  and  $\Psi_t$  in  $\text{End}(\mathcal{E})$  is zero, hence

$$t[\partial, \Psi_t] = [\Psi_t, A]$$

Moreover one has  $\partial \cdot \Psi_t = [\partial, \Psi_t]$  where  $\partial \cdot \Psi_t$  is the matrix of derivatives of the entries of  $\Psi_t$ . In particular, one gets for all  $n \geq 0$

$$t \text{Tr}(\Psi_t^n \partial \cdot \Psi_t) = \text{Tr}(\Psi_t^n [\Psi_t, A]) = \text{Tr}(\Psi_t^{n+1} A) - \text{Tr}(\Psi_t^n A \Psi_t) = 0.$$

If  $t = 0$ , then  $\Psi_0 = A^p$  and we immediately see that  $\text{Tr}(\Psi_0^n \partial \cdot \Psi_0) = 0$ . If  $t \neq 0$ , since  $R$  is integral, we get  $\text{Tr}(\Psi_t^n \partial \cdot \Psi_t) = 0$ .

Now we apply the formula  $\partial \cdot \det M = \det M \text{Tr}(M^{-1} \partial \cdot M)$  to the matrix  $M = \text{Id} - T\Psi_t$  and we write  $M^{-1} = \sum_{n \geq 0} T^n \Psi_t^n$ ,

$$\partial \cdot \det(\text{Id} - T\Psi_t) = -T \det(\text{Id} - T\Psi_t) \sum_{n \geq 0} T^n \text{Tr}(\Psi_t^n \partial \cdot \Psi_t).$$

Since, for all  $t$  and  $n$ , the elements  $\text{Tr}(\Psi_t^n \partial \cdot \Psi_t)$  are zero, we get the result. ■

## 4 The stack of nilpotent connections

We consider the embedding  $\mathbb{A}^1 \hookrightarrow W \times \mathbb{A}^1$  by the zero section and we denote by  $\mathcal{N}ilp(r, X)$  the corresponding fibre product  $\mathcal{C}(r, X) \times_{W \times \mathbb{A}^1} \mathbb{A}^1$ ,

$$\begin{array}{ccc} \mathcal{N}ilp(r, X) & \longrightarrow & \mathcal{C}(r, X) \\ \downarrow \text{H} & & \downarrow \text{H} \\ \mathbb{A}^1 & \xrightarrow{0} & W \times \mathbb{A}^1 \end{array}$$

The category  $\mathcal{N}ilp(r, X)(S)$  over a  $t$ -scheme  $S$  is the category of  $t$ -bundles  $(\mathcal{E}, \nabla_t)$  over  $X \times S$  such that  $(\Psi_t(\nabla_t))^r \in \text{Hom}(\mathcal{E}, \mathcal{E} \otimes q^* \omega_X^{pr})$  is zero. We say that  $\nabla_t$  is nilpotent. The stack of nilpotent connections is far from being reduced if  $r > 1$ . We define the exponent of nilpotence of the  $t$ -connection  $\nabla_t$  as the smallest integer  $l \geq 1$  such that  $(\Psi_t(\nabla_t))^l = 0$  and we denote by  $\mathcal{N}ilp_l(r, X)$  the substack of  $\mathcal{N}ilp(r, X)$  corresponding to nilpotent  $t$ -connections of exponent  $\leq l$ . Thus we get a filtration

$$\mathcal{N}ilp_1(r, X) \subset \cdots \subset \mathcal{N}ilp_r(r, X) = \mathcal{N}ilp(r, X)$$

and by Theorem 5.1 [K] we have an isomorphism induced by the relative Frobenius  $F_X$

$$(4.1) \quad \mathcal{M}(r, X^{(p)}) \xrightarrow{\sim} \mathcal{N}ilp_1(r, X), \quad E \mapsto (F_X^* E, \nabla^{can}).$$

### 4.1

Following [Si], we say that  $(\mathcal{E}, \nabla_t)$  over  $X \times S$  (or simply  $\nabla_t$ ) is semi-stable if for all geometric points  $\bar{s}$  of  $s \in S$ , any proper  $\nabla_{t, \bar{s}}$ -invariant subsheaf of  $\mathcal{E}_{\bar{s}}$  has negative degree (recall that  $\mathcal{E}$  is of degree 0). We denote by  $\mathcal{C}^{ss}(r, X)$  (resp.  $\mathcal{N}ilp_l^{ss}(r, X)$ ) the substack of  $\mathcal{C}(r, X)$  (resp.  $\mathcal{N}ilp_l(r, X)$ ) parameterizing semi-stable  $t$ -connections (resp. semi-stable  $t$ -connections which are nilpotent of exponent  $\leq l$ ).

**Lemma 4.1** *The natural morphism*

$$\mathcal{C}^{ss}(r, X) \rightarrow \mathcal{C}(r, X)$$

*is an open immersion.*

*Proof:* Let  $(\mathcal{E}, \nabla_t)$  be an  $S$ -point of  $\mathcal{C}(r, X)$ . One has to prove that the semi-stable sublocus in  $S$  is open. Because  $\mathcal{C}(r, X)$  is locally of finite type, one can assume that  $S$  is of finite type. By the standard argument for boundedness, there exists  $n \gg 0$  such that every subsheaf of  $\mathcal{E}_s$ ,  $s \in S(k)$ , has slope  $\leq n$ . Let  $I$  be the finite set

$$I = \{(r, d) \in \{1, \dots, r-1\} \times \mathbb{Z} \text{ such that } 0 < \frac{d}{r} \leq n\}.$$

Let  $\mathcal{Q}^I$  be the relative Hilbert scheme parameterizing subsheaves  $F_s$  of  $\mathcal{E}_s$  with Hilbert polynomial in  $I$ . Because  $I$  is finite,  $\mathcal{Q}^I$  is proper over  $S$ . But  $F_s$  is flat (i.e.  $\nabla_{t, s}$ -invariant) if and only if the linear morphism  $F_s \rightarrow (\mathcal{E}_s/F_s) \otimes \omega$  is zero. As in the proof of Proposition 3.1, it follows that the sublocus of  $\mathcal{Q}^I$  parameterizing flat subsheaves is closed. Its image in  $S$  is closed and is exactly the locus where  $(\mathcal{E}_s, \nabla_{t, s})$  is not semi-stable. ■

## 4.2

Let  $E$  be a vector bundle on  $X$  with Harder-Narasimhan filtration

$$0 = E_0 \subsetneq E_1 \subsetneq \dots \subsetneq E_l = E.$$

We denote by  $\mu_+(E)$  (resp.  $\mu_-(E)$ ) the slope of the semi-stable bundle  $E_1/E_0$  (resp.  $E_l/E_{l-1}$ ). One has by construction

$$\mu_-(E) \leq \mu(E) \leq \mu_+(E)$$

with equality if and only if  $l = 1$  and  $E$  semi-stable.

**Lemma 4.2** *The algebraic stack  $\mathcal{C}^{ss}(r, X)$  is of finite type.*

*Proof:* By the boundedness of the family of semi-stable bundles of given slope and rank, it is enough to find a bound for  $\mu_+(E) - \mu_-(E)$  where  $(E, \nabla_t)$  is semi-stable on  $X$ . Assume  $l > 1$  and let us consider the linear morphism

$$\bar{\nabla}_i : E_i \rightarrow (E/E_i) \otimes \omega_X$$

induced by  $\nabla_t$  for  $1 < i < l$ . By construction,  $\mu(E_i) > \mu(E) = 0$ . If  $\bar{\nabla}_i$  vanishes,  $E_i$  is flat and therefore of non positive slope, a contradiction. It follows that

$$\mu(E_i/E_{i-1}) = \mu_-(E_i) \leq \mu_+((E/E_i) \otimes \omega_X) = \mu(E_{i+1}/E_i) + 2g - 2.$$

In particular, one has

$$0 \leq \mu_+(E) - \mu_-(E) \leq (l-1)(2g-2) \leq (r-1)(2g-2).$$

■

## 5

We give a properness property of the deformation of the nilpotent cone of Higgs bundles induced by  $H$ . The precise statement is

**Proposition 5.1** *For all  $l \geq 1$ , the morphism of stacks of finite type*

$$H : \mathcal{N}ilp_l^{ss}(r, X) \rightarrow \mathbb{A}^1$$

*is universally closed.*

**Remark 5.1** We recall that, if  $l = 1$ , the fibres  $H^{-1}(0)$  and  $H^{-1}(1)$  are equal to the moduli stacks  $\mathcal{M}^{ss}(r, X)$  and  $\mathcal{M}^{ss}(r, X^{(p)})$  of semi-stable vector bundles, respectively (see (4.1)).

By [LM], it is enough to prove the following semi-stable reduction theorem.

**Proposition 5.2** *Let  $R$  be a complete discrete valuation ring with field of fractions  $K$  and  $t \in R$ . Let  $\mathcal{X}$  be a smooth, projective, connected curve over  $R$  and  $(\mathcal{E}_K, \nabla_K)$  a semi-stable  $t$ -bundle on  $\mathcal{X}_K$  with nilpotent  $p$ -curvature of exponent  $l$ . Then there exists a semi-stable  $t$ -bundle  $(\mathcal{E}, \nabla)$  with nilpotent  $p$ -curvature on  $\mathcal{X}$  extending  $(\mathcal{E}_K, \nabla_K)$  and of exponent  $l$ .*



*Proof:* We adapt Faltings's proof, based on ideas of Langton, of the properness of the moduli of Higgs bundles ([F] Theorem I.3) to  $t$ -bundles. If  $t = 0$ , the result is a particular case of Faltings's theorem. So we assume  $t \neq 0$ . We denote by  $F : \mathcal{X} \rightarrow \mathcal{X}^{(p)}$  the relative Frobenius, by  $F_K : \mathcal{X}_K \rightarrow \mathcal{X}_K^{(p)}$  its restriction to the generic fiber, and by  $\Psi_K$  the  $p$ -curvature of the connection  $\nabla_K/t$  on  $\mathcal{E}_K$ . By assumption  $\Psi_K^l = 0$ .

**Lemma 5.1** *Suppose that  $(\mathcal{E}_K, \nabla_K)$  is a (not necessarily semi-stable)  $t$ -bundle of exponent of nilpotence  $l$  on  $\mathcal{X}_K$ . Then it can be extended to the whole  $\mathcal{X}$  with exponent  $l$ .*

*Proof of the lemma:* We proceed by induction on  $l$ . If  $l = 1$ ,  $\Psi_K = 0$  and, by Theorem 5.1 [K], there exists a vector bundle  $\mathcal{E}_K^{(p)}$  on  $\mathcal{X}_K^{(p)}$  such that  $F_K^* \mathcal{E}_K^{(p)} \xrightarrow{\sim} \mathcal{E}_K$  and  $\nabla_K/t$  becomes the canonical connection  $\nabla_K^{can}$  on  $F_K^* \mathcal{E}_K^{(p)}$ . Now  $\mathcal{E}_K^{(p)}$  extends to a vector bundle  $\mathcal{E}^{(p)}$  on  $\mathcal{X}^{(p)}$  and the  $t$ -bundle  $(F^* \mathcal{E}^{(p)}, t \nabla^{can})$  extends  $(\mathcal{E}_K, \nabla_K)$ , since  $t \in R$ .

Now assume that  $l > 1$  and that the lemma holds for  $t$ -bundles with exponent of nilpotence  $< l$ . Let  $F_K^n$  be the kernel of  $\Psi_K^n$  for  $n = 1, \dots, l$ . Thus we have a filtration

$$0 = F_K^0 \subset F_K^1 \subset \dots \subset F_K^{l-1} \subset F_K^l = \mathcal{E}_K.$$

Since  $\Psi_K$  and  $\nabla_K/t$  commute (Proposition 5.2 [K]), the connection  $\nabla_K/t$  maps  $F_K^n$  to  $F_K^n \otimes \omega_{\mathcal{X}_K/K}$ . By construction, the induced connection  $\nabla_K^F$  on  $F_K^{l-1}$  is of exponent  $l-1$ . Moreover  $\nabla_K/t$  induces a connection  $\nabla_K^{gr}$  on the quotient  $gr_K := \mathcal{E}_K / F_K^{l-1}$ , which is of exponent 1, since  $\Psi_K$  maps  $\mathcal{E}_K$  to  $F_K^{l-1}$ .

By induction the  $t$ -bundles  $(F_K^{l-1}, \nabla_K^F)$  and  $(gr_K, \nabla_K^{gr})$  have models on  $\mathcal{X}$ , which we denote by  $(F^{l-1}, \nabla^F)$  and  $(gr, \nabla^{gr})$ , respectively.

Let  $x$  be the generic point of the special fiber of  $\mathcal{X}$ . The  $R$ -algebra  $A = \mathcal{O}_{\mathcal{X},x}$  is a discrete valuation ring with field of fraction  $L = k(\eta)$  where  $\eta$  is the generic point of  $\mathcal{X}$ . By Proposition 6 [L], it is enough to find an  $A$ -lattice  $E_x$  in the  $r$ -dimensional  $L$ -vector space  $\mathcal{E}_{K,\eta}$  such that

$$(5.1) \quad \nabla_K/t(E_x) \subset E_x \otimes_A \omega_x$$

where  $\omega_x = \omega_{\mathcal{X}/R,x}$ .

Now,  $\mathcal{E}_K$  is an extension of  $gr_K$  by  $F_K^{l-1}$  whose extension class  $e_K$  lives in

$$H^1(\mathcal{X}_K, \text{Hom}(gr_K, F_K^{l-1})) = H^1(\mathcal{X}, \text{Hom}(gr, F^{l-1})) \otimes K.$$

Let  $\pi$  be a uniformizing parameter of  $R$ . If  $m$  is large enough,  $\pi^m e_K$  defines an extension

$$(5.2) \quad 0 \rightarrow F^{l-1} \rightarrow \mathcal{E} \rightarrow gr^l \rightarrow 0$$

on  $\mathcal{X}$  together with an isomorphism

$$\mathcal{E} \otimes K \xrightarrow{\sim} \mathcal{E}_K.$$

The localization of (5.2) at  $x$  is split ( $A$  is principal) and, with respect to this splittings,  $\nabla_K/t$  is of the form

$$\nabla_K/t = \begin{pmatrix} \nabla^F & 0 \\ 0 & \nabla^{gr} \end{pmatrix} + \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$$

where  $M \in L \otimes \text{Hom}(gr_x, F_x^{l-1} \otimes \omega_x)$ . Let  $m$  be an integer such that  $\pi^m M \in \text{Hom}(gr_x, F_x^{l-1} \otimes \omega_x)$  and let  $\mathcal{E}_x^m \subset \mathcal{E}_{K,\eta}$  be the lattice defined by the pull-back of

$$0 \rightarrow F_x^{l-1} \rightarrow \mathcal{E}_x \rightarrow gr_x \rightarrow 0$$

by

$$\begin{cases} \mathrm{gr}_x & \rightarrow & \mathrm{gr}_x \\ g & \mapsto & \pi^m g \end{cases}$$

Hence, by construction, the  $\Lambda$ -lattice  $\mathcal{E}_x^m$  satisfies (5.1). Then the vector bundle  $\mathcal{E}^m$  defined by  $\mathcal{E}_K$  and  $\mathcal{E}_x^m$ , and the  $t$ -connection  $\nabla_K = t\nabla_K/t$  define a  $t$ -bundle extending  $(\mathcal{E}_K, \nabla_K)$  of exponent  $l$ . ■

The rest of the proof goes exactly as in [F] (or [L]). Let us for the convenience of the reader recall the argument. Let  $s$  be the closed point of  $\mathrm{Spec}(\mathbf{R})$  and  $\bar{s}$  a geometric point over it. As usual every  $t_{\bar{s}}$ -bundle over  $\mathcal{X}_{\bar{s}}$  has a unique flat (i.e., invariant) subbundle of maximal slope. From this uniqueness, it follows easily (adapt the arguments of [S] where the case  $t_s = 0$  is treated, for instance), this bundle is in fact defined over  $\mathcal{X}_s$ . In particular, there is a notion of Harder-Narasimhan filtration of  $t_s$ -bundles. The set of slopes of the first term  $F^\alpha(\mathcal{E}_s, \nabla_s)$  of the Harder-Narasimhan filtration of the  $t_s$ -bundle  $(\mathcal{E}_s, \nabla_s)$  where  $(\mathcal{E}, \nabla)$  runs over the models of  $(\mathcal{E}_K, \nabla_K)$  with  $p$ -curvature of exponent  $l$ , whose existence is proved in Lemma 5.1, has certainly a smallest element  $\alpha$ . Among these models of minimal slopes  $\alpha$ , pick-up one,  $(\mathcal{E}^0, \nabla^0)$  say, such that  $F^\alpha = F^\alpha \mathcal{E}_s^0$  has minimal rank. We will show that the  $t$ -bundle  $(\mathcal{E}^0, \nabla^0)$  is a semi-stable model. Suppose that the contrary holds, i.e.,  $\mathrm{rk}(F^\alpha) < r$ .

The connection  $\nabla^0$  induces a connection  $\nabla^1$  on the kernel  $\mathcal{E}^1$  of the surjection  $\mathcal{E}^0 \rightarrow \mathcal{E}_s^0/F^\alpha$  and therefore we get a new model  $(\mathcal{E}^1, \nabla^1)$  with  $\mathcal{E}^1$  locally free. Notice that  $\nabla^1$  still has  $p$ -curvature of exponent  $l$ . Because

$$\mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{E}_s^0/F^\alpha, \mathcal{O}_{\mathcal{X}_s}) = \mathcal{E}_s^0/F^\alpha,$$

the restriction of the exact sequence

$$0 \rightarrow \mathcal{E}^1 \rightarrow \mathcal{E}^0 \rightarrow \mathcal{E}_s^0/F^\alpha \rightarrow 0$$

to  $\mathcal{X}_s$  gives an exact sequence of flat bundles

$$0 \rightarrow \mathcal{E}_s^0/F^\alpha \rightarrow \mathcal{E}_s^1 \rightarrow F^\alpha \rightarrow 0.$$

Let  $F_1^\beta$  be the first term of the Harder-Narasimhan filtration of  $(\mathcal{E}_s^1, \nabla_{t,s})$ . By minimality of  $\alpha$ , one has  $\alpha \leq \beta$ . Observe that the Harder-Narasimhan filtration  $\mathcal{E}_s^0/F^\alpha$  has semi-stable subquotients of slopes  $< \alpha \leq \beta$ . Therefore, every morphism from a stable bundle of slope  $\beta$  to  $\mathcal{E}_s^0/F^\alpha$  is zero. It follows immediately that  $\beta = \alpha$  and that  $F_1^\beta$  injects to  $F^\alpha$ . By minimality of the rank of  $F^\alpha$ , this injection is an isomorphism. Therefore  $F^\alpha$  is a subsheaf of  $\mathcal{E}^1$ . By induction, one can construct the subsheaf  $\mathcal{E}^{n+1}$  of  $\mathcal{E}^0$  as the kernel of  $\mathcal{E}^n \rightarrow \mathcal{E}_s^n/F^\alpha$  with a  $(t \bmod \pi^{n+1})$ -connection  $\nabla^{n+1}$  induced by  $\nabla^n$ . The point is that one has the exact sequence (of subsheaves of  $\mathcal{E}^0$ )

$$0 \rightarrow \pi^n \mathcal{E}^n \rightarrow \mathcal{E}^{n+1} \rightarrow F^\alpha \rightarrow 0.$$

In particular,  $\pi^n \mathcal{E}^0$  is contained in  $\mathcal{E}^n$  and  $\mathcal{E}^n/\pi^n \mathcal{E}^0$  injects in  $\mathcal{E}^0/\pi^n \mathcal{E}^0$ . Taking the direct limit, we get a subsheaf of the completion of  $\mathcal{E}^0$  along  $\mathcal{X}_s$  with restricts to  $F^\alpha$  on the special fiber. It comes with a  $t$ -connection induced by  $\nabla^0$ , and both the sheaf (which is a vector bundle) and the  $t$ -connection algebraizes by properness of  $\mathcal{X} \rightarrow \mathrm{Spec}(\mathbf{R})$ . Moreover, this subsheaf is certainly  $\nabla$ -invariant. By flatness over  $\mathbf{R}$ , the slope of its generic fiber is  $\alpha > 0$ , contradicting  $(\mathcal{E}_K, \nabla_K)$  semi-stable. ■

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